Invariants for surface-links and virtual surface-links

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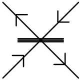
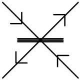
Outline

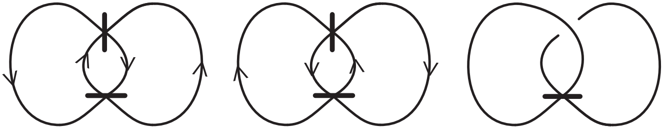
1. Marked graphs in $\mathbb{R}^3$ and their diagrams
2. Generalized Kauffman bracket for marked graphs
3. Surface-links and marked graphs
4. A specialization of the generalized Kauffman bracket → invariants for surface-links
5. Generalization to virtual marked graph diagrams

Marked graphs in 3-space

- A marked graph (shortly, MG) is a spatial graph G in $\mathbb{R}^3$ which satisfies the following
  - G is a finite regular graph possibly with 4-valent vertices, say $v_1, v_2, \ldots, v_n$.
  - Each v_i is a rigid vertex, i.e., we fix a sufficiently small rectangular neighborhood $N_i \cong \{(x,y) \in \mathbb{R}^2 | -1 \leq x, y \leq 1\}$, where v_i corresponds to the origin and the edges incident to v_i are represented by $x^2 = y^2$.
  - Each v_i has a marker, which is the thickened interval on N_i given by $\{(x,0) \in \mathbb{R}^2 | -\frac{1}{2} \leq x \leq \frac{1}{2}\}$, i.e.,  .
- Two marked graphs are said to be equivalent if they are ambient isotopic in $\mathbb{R}^3$ with keeping the rectangular neighborhoods and markers.

Oriented marked graphs

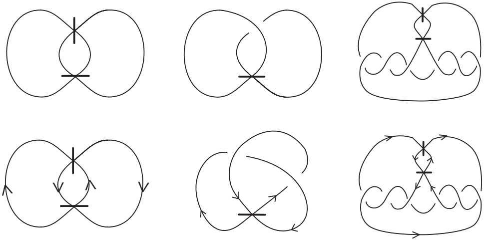
- An orientation of a marked graph G is a choice of an orientation for each edge of G in such a way that every vertex in G looks like  or  .
- A marked graph is said to be orientable if it admits an orientation. Otherwise, it is said to be non-orientable.
- An oriented marked graph means an orientable marked graph with a fixed orientation.



- In this talk, an unoriented marked graph means a non-orientable marked graph or an orientable marked graph without a fixed orientation.

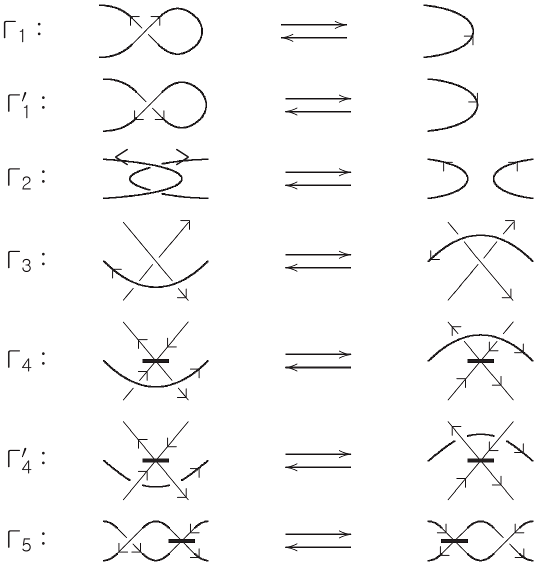
Marked graph diagrams

- An unoriented/oriented marked graph G in $\mathbb{R}^3$ can be described as usual by a diagram D in $\mathbb{R}^2$, which is an unoriented/oriented link diagram in $\mathbb{R}^2$ possibly with some marked 4-valent vertices.



- Two unoriented/oriented MG diagrams in $\mathbb{R}^2$ represent equivalent unoriented/oriented marked graphs in $\mathbb{R}^3$ if and only if they are transformed into each other by a finite sequence of the unoriented/oriented RV4 graph moves $\Gamma_1, \Gamma_1', \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_4'$ and Γ_5:

Oriented RV4 graph moves

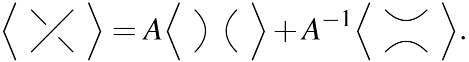


Kauffman bracket polynomial

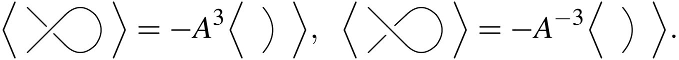
- Let K be a knot or link diagram. The Kauffman bracket polynomial of K is a Laurent polynomial $\langle K\rangle = \langle K\rangle(A) \in \mathbb{Z}[A, A^{-1}]$ defined by the following rules:

(B1) $\langle \bigcirc \rangle = 1$,

(B2) $\langle \bigcirc\ K' \rangle = \delta \langle K' \rangle$, where $\delta = -A^2 - A^{-2}$,

(B3)


- The Kauffman bracket polynomial is a regular isotopy invariant for unoriented links and

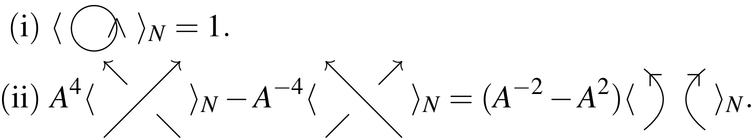


Normalized Kauffman bracket polynomial

- Let L be an oriented link diagram and let $\widetilde{L}$ be the link diagram L without orientation. The normalized Kauffman bracket polynomial $\langle L\rangle_N$ of L is defined by

$$\langle L\rangle_N = (-A^3)^{-w(L)}\langle\widetilde{L}\rangle$$

- The normalized Kauffman bracket polynomial is an invariant of the oriented link in $\mathbb{R}^3$ presented by L, and satisfies the recursive formula:



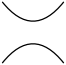
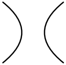
(i) $\langle \quad \rangle_N = 1.$

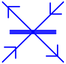
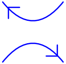
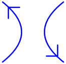
(ii) $A^4\langle \quad \rangle_N - A^{-4}\langle \quad \rangle_N = (A^{-2} - A^2)\langle \quad \rangle_N.$

Generalized Kauffman bracket $[[\ \ ]]$ for MG diagrams

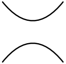
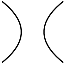
Let D be an unoriented/oriented MG diagram.
Let $[[D]]$ be the polynomial in $\mathbb{Z}[A,A^{-1}][x,y]$ defined by the following two rules:

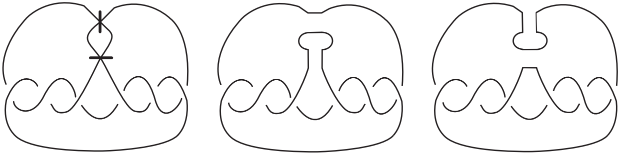
(L1) $[[D]] = \langle D\rangle / \langle D\rangle_N$ if D is an unoriented/oriented link diagram,

(L2) $[[$  $]] = x[[$  $]] + y[[$  $]].$

$[[$  $]] = x[[$  $]] + y[[$  $]].$

Resolutions of MG diagrams

- For an unoriented MG diagram D, let $L_-(D)$ and $L_+(D)$ be the oriented link diagrams obtained from D by replacing each marked vertex  with  or .



D　　L_(D)　　L+(D)

- We call $L_-(D)$ and $L_+(D)$ the negative resolution and the positive resolution of D, respectively.

Self-writhe of MG diagrams

- Let $D = D_1 \cup \cdots \cup D_m$ be an oriented link diagram and let $w(D_i)$ be the usual writhe of the component D_i. The self-writhe $sw(D)$ of D is defined to be the sum

$$sw(D) = \sum_{i=1}^{m} w(D_i).$$

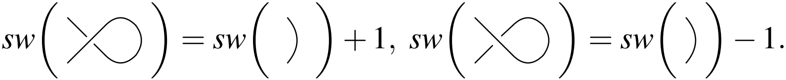
- Let D be an unoriented MG diagram. We choose an arbitrary orientation for each component of $L_+(D)$ and $L_-(D)$. Define the self-writhe $sw(D)$ of D by

$$sw(D) = \frac{sw(L_+(D)) + sw(L_-(D))}{2}.$$

where $sw(L_+(D))$ and $sw(L_-(D))$ are independent of the choice of orientations because the writhe $w(D_i)$ is independent of the choice of orientation for D_i.

Normalization of $[[\ \ ]]$

Let D be an unoriented MG diagram. Then $sw(D)$ is invariant under all RV4 graph moves except the unoriented move $\widetilde{\Gamma}_1$. For $\widetilde{\Gamma}_1$ and its mirror move,



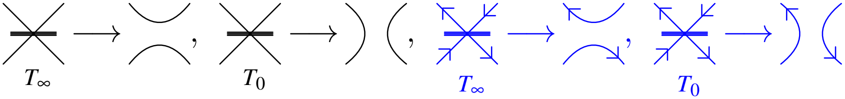
Definition (Generalized Kauffman bracket polynomial)

Let D be an unoriented/oriented MG diagram. We define $\ll D \gg$ /$\ll D \gg_N$ to be the polynomial in variables x and y with coefficients in $\mathbb{Z}[A, A^{-1}]$ given by

$$\ll D \gg = (-A^3)^{-sw(D)}[[D]](x,y) / \ll D \gg_N = [[D]](x,y).$$

State-sum formulas for $\ll \quad \gg$ and $\ll \quad \gg_N$

Let D be an unoriented/oriented marked graph diagram and let $V(D)$ be the set of all marked vertices. A state of D is a function $\sigma : V(D) \to \{+1, -1\}$, i.e., an assignment of $+1$ or -1 to each marked vertex of D. Let $\mathscr{S}(D)$ be the set of all states of D. For $\sigma \in \mathscr{S}(D)$, let D_σ denote the link diagram obtained from D by



Then

$$\ll D \gg = (-A^3)^{-sw(D)} \sum_{\sigma \in \mathscr{S}(D)} \prod_{v \in V(D)} x^{\frac{1+\sigma(v)}{2}} y^{\frac{1-\sigma(v)}{2}} \langle D_\sigma \rangle,$$

$$\ll D \gg_N = (-A^3)^{-w(D)} \sum_{\sigma \in \mathscr{S}(D)} \prod_{v \in V(D)} x^{\frac{1+\sigma(v)}{2}} y^{\frac{1-\sigma(v)}{2}} \langle \widetilde{D_\sigma} \rangle.$$

Polynomial invariant for marked graphs in 3-space

Theorem

Let D be an oriented MG diagram. Then

$$\ll D \gg_N = (-A^3)^{sw(\widetilde{D})-w(D)} \ll \widetilde{D} \gg .$$

Theorem

Let G be an unoriented/oriented marked graph in $\mathbb{R}^3$ and let D be an unoriented/oriented MG diagram presenting G. Then the polynomial $\ll D \gg / \ll D \gg_N$ is an invariant for unoriented/oriented RV4 graph moves, and therefore it is an invariant of G.

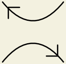
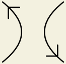
Recursive formula for $\ll D \gg_N$

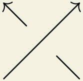
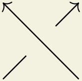
Theorem

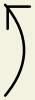
(1) $\ll$  $\gg_N = 1.$

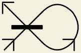
(2) *If D and D' are two equivalent oriented MG diagrams, then* $\ll D \gg_N = \ll D' \gg_N.$

(3) $\ll D \sqcup$  $\gg_N = (-A^{-2} - A^{2}) \ll D \gg_N.$

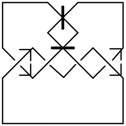
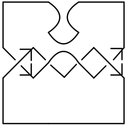
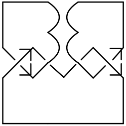
(4) $\ll$  $\gg_N = x \ll$  $\gg_N + y \ll$  $\gg_N.$

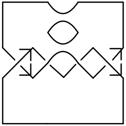
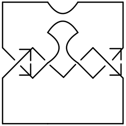
(5) $A^4 \ll$  $\gg_N - A^{-4} \ll$  $\gg_N = (A^{-2} - A^{2}) \ll$  $\gg_N.$

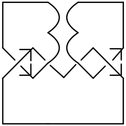
(6) $\ll$  $\gg_N = \left(y - (A^{-2} + A^{2})x\right) \ll$  $\gg_N.$

(7) $\ll$  $\gg_N = \left(x - (A^{-2} + A^{2})y\right) \ll$  $\gg_N.$

Example

$\ll$  $\gg_N = x^2 \ll$  $\gg_N + xy \ll$  $\gg_N +$

$yx \ll$  $\gg_N + y^2 \ll$  $\gg_N$

$= \langle \bigcirc \bigcirc \rangle_N (x^2 + y^2) + xy \langle$  $\rangle_N + yx \langle \bigcirc \bigcirc \bigcirc \rangle_N$

$$= (-A^{-2} - A^2)(x^2 + y^2) + (-A^{10} - A^2)(-A^{-10} - A^{-2})xy + (-A^{-2} - A^2)^2 xy$$

$$= (-A^2 - A^{-2})(x^2 + y^2) + (A^4 + 4 + A^{-4} + A^{-8} + A^8)xy.$$

Some properties of $\ll D \gg_N$

Theorem

(1) $\ll D \gg_N = \ll -D \gg_N$.

(2) $\ll D^* \gg_N(A,x,y) = \ll D \gg_N(A,y,x)$.

(3) $\ll D! \gg_N(A,x,y) = \ll D \gg_N(A^{-1},x,y)$.

(4) $\ll D\sharp D' \gg_N = \ll D \gg_N \ll D' \gg_N$.

(5) $\ll D \sqcup D' \gg_N = (-A^2 - A^{-2}) \ll D \gg_N \ll D' \gg_N$.

(6) $\ll D * D' \gg_N = \left(x^2 - 2(A^2 + A^{-2})xy + y^2\right) \ll D \gg_N \ll D' \gg_N$.

(a) $D\sharp D'$ (b) $D * D'$

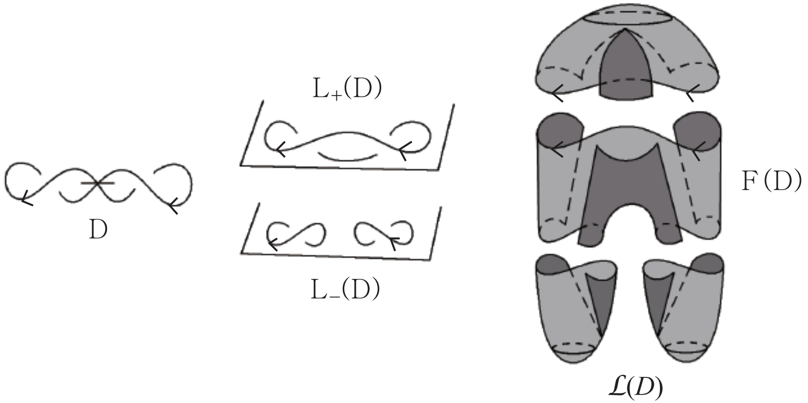
Surface-links

- A surface-link is a closed surface smoothly embedded in $\mathbb{R}^4$ (or in S^4).
- A connected surface-link is called a surface-knot.
  · A 2-sphere-link is sometimes called a 2-link.
  · A connected 2-link is called a 2-knot.
- Two surface-links $\mathscr{L}$ and $\mathscr{L}'$ in $\mathbb{R}^4$ are equivalent if they are ambient isotopic, i.e.,
  $\exists$ orient. pres. homeo. $h : \mathbb{R}^4 \to \mathbb{R}^4$ s.t. $h(\mathscr{L}) = \mathscr{L}'$.
- If each component $\mathscr{K}_i$ of a surface-link $\mathscr{L} = \mathscr{K}_1 \cup \cdots \cup \mathscr{K}_\mu$ is oriented, $\mathscr{L}$ is called an oriented surface-link. Two oriented surface-links $\mathscr{L}$ and $\mathscr{L}'$ are equivalent if the restriction $h|_{\mathscr{L}} : \mathscr{L} \to \mathscr{L}'$ is also orientation preserving.

Adm. MG diagram $D \longrightarrow$ Surface-link $\mathscr{L}(D)$

Definition

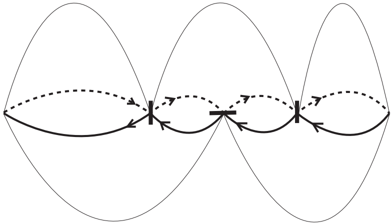
A MG diagram D is admissible if both resolutions $L_-(D)$ and $L_+(D)$ are trivial link diagrams.



Surface-links $\longrightarrow$ adm. MG diagrams

Any surface link $\mathscr{L}$ in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ can be deformed into a surface link $\mathscr{L}'$, called a hyperbolic splitting of $\mathscr{L}$, by an ambient isotopy of $\mathbb{R}^4$ in such a way that the projection $p : \mathscr{L}' \to \mathbb{R}$ satisfies the followings:

- all critical points are non-degenerate,
- all the index 0 critical points (minimal points) are in $\mathbb{R}^3_{-1}$,
- all the index 1 critical points (saddle points) are in $\mathbb{R}^3_0$,
- all the index 2 critical points (maximal points) are in $\mathbb{R}^3_1$.



Surface-links $\longrightarrow$ adm. MG diagrams

- Then the cross-section $\mathscr{L}'_0 = \mathscr{L}' \cap \mathbb{R}^3_0$ at $t = 0$ is a spatial 4-valent regular graph in $\mathbb{R}^3_0$. We give a marker at each 4-valent vertex (saddle point) that indicates how the saddle point opens up above as illustrated in Figure:

- When $\mathscr{L}$ is an oriented surface-link, we choose an orientation for each edge of $\mathscr{L}'_0$ so that it coincides with the induced orientation on the boundary of $\mathscr{L}' \cap \mathbb{R}^3 \times (-\infty, 0]$ by the orientation of $\mathscr{L}'$ inherited from the orientation of $\mathscr{L}$.
- The resulting marked graph $G := \mathscr{L}'_0$ is called an marked graph presenting $\mathscr{L}$ and its diagram D (admissible) is called a marked graph diagram presenting $\mathscr{L}$.

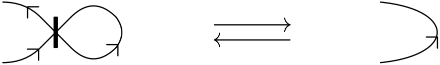
Surface-links & Adm. MG diagrams

Theorem (Kearton-Kurlin, Swenton)

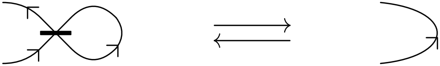
Two unoriented/oriented admissible marked graph diagrams present the same unoriented/oriented surface-link if and only if they are transformed into each other by a finite sequence of unoriented/oriented RV4 graph moves (called unoriented/oriented Yoshikawa moves of type I) and unoriented/oriented Yoshikawa moves of type II:

Oriented Yoshikawa moves of type II

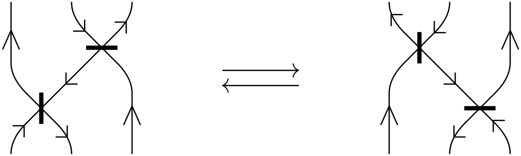
Γ_6 :



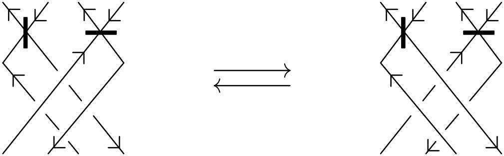
Γ_6' :



Γ_7 :



Γ_8 :



A specialization of $\ll \quad \gg$ and $\ll \quad \gg_N$

Let

$$z(t) = \frac{1}{2\sqrt{t}}\left(\sqrt{3t-1} + \mathbf{i}\sqrt{t+1}\right), \;\; \bar{z}(t) = \frac{1}{2\sqrt{t}}\left(\sqrt{3t-1} - \mathbf{i}\sqrt{t+1}\right),$$

where $t \neq 0$ and $\mathbf{i} = \sqrt{-1}$. Note that $\bar{z}(t) = z(t)^{-1}$.

Definition

Let D be an unoriented/oriented marked graph diagram. We define $\mathbf{K}(D)/\mathbf{K}(D)_N$ by the formula:

$$\begin{aligned} \mathbf{K}(D) = \mathbf{K}(D;t) &= \ll D \gg |_{A=z(t),\ A^{-1}=\bar{z}(t),\ x=y=t} \\ &= (-z(t)^3)^{-sw(D)}[[D]](z(t), \bar{z}(t), t, t). \\ \mathbf{K}(D)_N = \mathbf{K}(D;t)_N &= \ll D \gg |_{A=z(t),\ A^{-1}=\bar{z}(t),\ x=y=t} \\ &= (-z(t)^3)^{-w(D)}[[\widetilde{D}]](z(t), \bar{z}(t), t, t). \end{aligned}$$

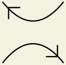
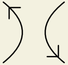
Recursive rules for $\mathbf{K}(D)_N$

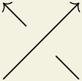
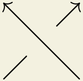
Theorem

(1) $\mathbf{K}($  $)_N = 1.$

(2) *If* $D \approx_{MG} D'$, *then* $\mathbf{K}(D)_N = \mathbf{K}(D')_N.$

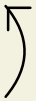
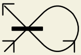
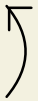
(3) $\mathbf{K}(D \sqcup$  $)_N = (t^{-1} - 1)\mathbf{K}(D)_N.$

(4) $\mathbf{K}\Big($  $\Big)_N = t\Big[\mathbf{K}\Big($  $\Big)_N + \mathbf{K}\Big($  $\Big)_N\Big].$

(5) $\lambda(t)\mathbf{K}\Big($  $\Big)_N - \overline{\lambda}(t)\mathbf{K}\Big($  $\Big)_N =$

$2\mathbf{i}t\sqrt{t+1}\sqrt{3t-1}\ \mathbf{K}\Big($  $\Big)_N$, *where*

$\lambda(t) = (t^2 + 2t - 1) - \mathbf{i}(t-1)\sqrt{t+1}\sqrt{3t-1}.$

(6) $\mathbf{K}\Big($  $\Big)_N = \mathbf{K}\Big($  $\Big)_N$, $\mathbf{K}\Big($  $\Big)_N = \mathbf{K}\Big($  $\Big)_N.$

Invariants for surface-links

Theorem

Let D be an unoriented/oriented marked graph diagram and let D' be an unoriented/oriented marked graph diagram obtained from D by applying a single unoriented/oriented Yoshikawa move. Then

$$\mathbf{K}(D') = \mathbf{K}(D) + (2t-1)\Psi(t) / \mathbf{K}(D')_N = \mathbf{K}(D)_N + (2t-1)\Psi(t),$$

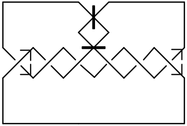
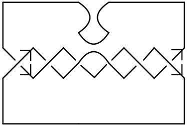
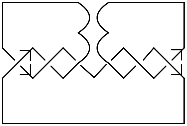
where $\Psi(t) \in \mathscr{M} = \mathbb{Z}[2^{-1}, t^{\frac{1}{2}}, t^{-\frac{1}{2}}, \sqrt{3t-1}, \mathbf{i}\sqrt{t+1}]$.

Corollary

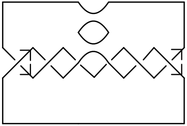
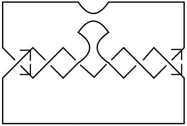
Let $\mathscr{L}$ be an unoriented/oriented surface-link and let D be an unorinted/oriented marked graph diagram presenting $\mathscr{L}$. Then $\mathbf{K}(D) + < 2t-1 > / \mathbf{K}(D)_N + < 2t-1 >$ *is an invariant of $\mathscr{L}$.*

Example

Let 8_1 be the spun 2-knot of the trefoil knot. Then

$\ll$  $\gg_N = x^2 \ll$  $\gg_N + xy \ll$  $\gg$

8_1

$yx \ll$  $\gg_N + y^2 \ll$  $\gg$

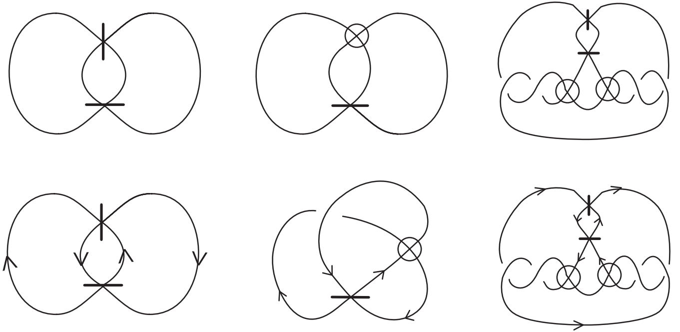
$$= (-A^{-2} - A^2)(x^2 + y^2) + (-A^{-2} - A^2)^2 xy +$$

$$(-A^{16} + A^{12} + A^4)(-A^{-16} + A^{-12} + A^{-4})xy$$

$$= (-A^2 - A^{-2})(x^2 + y^2) + (5 - A^{12} - A^{-12} + A^8 + A^{-8})xy.$$

$\mathbf{K}(8_1)_N = t^{-4}(6t^5 + 14t^4 - 8t^3 - 8t^2 + 6t - 1)$ and hence $t^{-4}(6t^5 + 14t^4 - 8t^3 - 8t^2 + 6t - 1) + < 2t - 1 >$ is an invariant of 8_1.

Virtual marked graph (VMG) diagrams



Equivalence of VMG diagrams

(1) Two oriented VMG diagrams D and D' are equivalent if they are transformed into each other by a finite sequence of the following oriented VMG moves:

- The moves $\Gamma_1, \ldots, \Gamma_5, -\Gamma_1$ and Γ'_4.
- The moves $V\Gamma_1, \ldots, V\Gamma_5$, $V\Gamma'_4$ and $-V\Gamma'_4$ below.

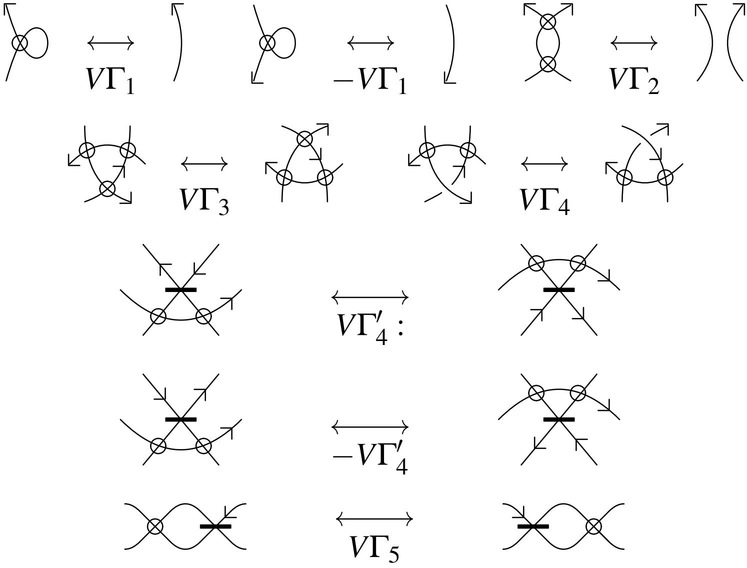
An oriented virtual marked graph is defined to be an equivalence class of oriented VMG diagrams modulo oriented VMG moves.

(2) Two VMG diagrams D and D' are said to be equivalent if they are transformed into each other by a finite sequence of the following VMG moves:

- The moves $\Omega_1, \ldots, \Omega_5$ and Ω'_4.
- The moves $V\Omega_1, \ldots, V\Omega_5$ and $V\Omega'_4$, where $V\Omega'_4$ and $V\Omega_5$ stand for the move $V\Gamma'_4$ and $V\Gamma_5$ forgetting the orientations.

A virtual marked graph is defined to be an equivalence class of VMG diagrams modulo VMG moves.

Oriented VMG moves



Virtual surface-links

Oriented generalized Yoshikawa moves:

- The oriented Yoshikawa moves $\Gamma_1, \ldots, \Gamma_5, -\Gamma_1$ and Γ_4' of Type I.
- The oriented virtual marked graph moves $V\Gamma_1, \ldots, V\Gamma_5$, $V\Gamma_4'$ and $-V\Gamma_4'$.
- The oriented Yoshikawa moves $\Gamma_6, \Gamma_6', \Gamma_7$ and Γ_8 of type II.

Definition

A virtual surface-link is defined to be an equivalence class of admissible VMG diagrams modulo generalized Yoshikawa moves. An oriented virtual surface-link is defined to be an equivalence class of oriented admissible VMG diagrams modulo unoriented generalized Yoshikawa moves.

Invariants for virtual surface-links

Theorem

Let D be an unoriented/oriented VMG diagram and let D' be an unoriented/oriented VMG diagram obtained from D by applying a single generalized unoriented/oriented Yoshikawa move. Then

$$\mathbf{K}(D') = \mathbf{K}(D) + (2t-1)\Psi(t) / \mathbf{K}(D')_N = \mathbf{K}(D)_N + (2t-1)\Psi(t),$$

where $\Psi(t) \in \mathscr{M} = \mathbb{Z}[2^{-1}, t^{\frac{1}{2}}, t^{-\frac{1}{2}}, \sqrt{3t-1}, \mathbf{i}\sqrt{t+1}]$.

Corollary

Let $\mathscr{L}$ be an unoriented/oriented virtual surface-link and let D be an unorinted/oriented VMG diagram presenting $\mathscr{L}$. Then $\mathbf{K}(D) + < 2t-1 > / \mathbf{K}(D)_N + < 2t-1 >$ *is an invariant of $\mathscr{L}$.*

Thank you!